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The heat equation and analytic continuation: Ivar Fredholm's first paper

D. Khavinson and H.S. Shapiro

1. Prologue

In the paper [F] (his first), Fredholm claims to have proven the following theorem: *the power series*

$$f(\zeta;a) = \sum_{0}^{\infty} a^n \zeta^{n^2},$$

where $a : 0 < a < 1$ is a parameter, has the following properties:

(i) *its radius of convergence is equal to 1, so f is holomorphic in the unit disk $\mathbb{D}$;*

(ii) *f and all its derivatives extend continuously to $\overline{\mathbb{D}}$;*

(iii) *f is not analytically continuable across any point of $\partial\mathbb{D}$ = the unit circle.*

This paper, although written in Swedish, became fairly well known because Fredholm's teacher, Mittag-Leffler, wrote a letter in French to Poincaré in which he described (and amplified) Fredholm's reasoning, and this letter was published in Acta Mathematica [ML]. It has since become a standard reference (cf. [B, remark following Thm. 4.3.2]) as one of the first (published) examples of a gap series representing a $C^\infty(\overline{\mathbb{D}})$-function, non-continuable across any point of the circle of convergence. We remark in passing that examples of this type can also be obtained by a suitable choice of coefficients in $\sum_0^\infty c_n \zeta^{2^n}$ (Weierstrass' series, although with regard to *noncontinuability* this example is sometimes attributed to Poincaré), and were thus readily constructible with the tools then known (cf. §6, below).

Let us discuss statements (i)–(iii) in more detail. (i) is obvious (and in fact holds for all $a \in \mathbb{C} \setminus \{0\}$) and so is (ii), provided $|a| < 1$. Moreover, (iii) is of course true as well, being a special case of a later theorem of Fabry (cf. [D., p. 376], or [B, §2 ff.]): If $\sum_0^\infty c_n \zeta^{\lambda_n}$ is a power series with a radius of convergence (r.o.c.) 1 and $\lambda_n / n \to \infty$, then the series is not continuable across any point of $\partial\mathbb{D}$. (Thus the choice of coefficients $c_n = a^n$ in Fredholm's example is not crucial for (iii), provided that the radius of convergence is 1.)

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Note that an earlier gap theorem of Hadamard (1892) does not apply to Fredholm's example, since that requires much larger gaps: $\lambda_{n+1}/\lambda_n \geq c > 1$.

2. Critique of Fredholm's proof.

Unfortunately, Fredholm's argument, based on a beautiful idea involving partial differential equations (namely, the heat equation in the complex domain), is flawed. The purpose of the present paper is to give a correct proof by suitably modifying Fredholm's reasoning, and to expound some of its ramifications.

He argued as follows: Set $a = e^z$ and $\zeta = e^w$, so the series becomes

$$u(z,w) := \sum_{n=0}^{\infty} e^{nz+n^2 w}. \tag{2.1}$$

This converges and defines a holomorphic function for $(z,w) \in \mathbb{C} \times \mathcal{L}$, where $\mathcal{L} := \{w \in \mathbb{C} : \operatorname{Re} w < 0\}$. Fredholm wants to show that for each $z_0 \in \mathbb{C}$,

$$w \mapsto \sum_{n=0}^{\infty} e^{nz_0} e^{n^2 w}$$

is not continuable across any point of $\partial\mathcal{L} := \{w : \operatorname{Re} w = 0\}$. Assuming it *were* continuable across $w_0 \in \partial\mathcal{L}$, he cites a theorem of Kovalevskaya [K] from her *Habilitationsschrift*, which (he says) implies that if u satisfies the "heat equation"

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial w} \tag{2.2}$$

(as does function (2.1)), and for a fixed z_0 extends analytically (as a function of w) to a neighborhood of $w_0 \in \partial\mathcal{L}$, then $z \mapsto \sum_{n=0}^{\infty} e^{n^2 w_0} e^{nz}$ extends from a neighborhood of z_0 to the whole complex z-plane, i.e., is an entire function of z. This gives Fredholm the desired contradiction because, in terms of the variable $a = e^z$, $u(z, w^0)$ is a power series with radius of convergence 1, so it certainly has singularities on the imaginary z-axis. But the argument is erroneous! Indeed, consider the function

$$v(z,w) := \sum_{n=0}^{\infty} \frac{(\sinh nz)}{R^n} e^{n^2 w}, \tag{2.3}$$

where $R > 1$ is fixed. (2.3) converges for $(z,w) \in \mathbb{C} \times \mathcal{L}$ and satisfies (2.2) there. For $z = z_0 = 0$, $v(z_0, w) \equiv 0$, so it is continuable across every point of $\partial\mathcal{L}$. Therefore, by Fredholm's reasoning for each $w_0 \in \partial\mathcal{L}$ (i.e., with $\operatorname{Re} w_0 = 0$), the function

$$z \mapsto \sum_{n=0}^{\infty} \frac{(e^{n^2 w_0}) \cdot \sinh nz}{R^n}, \qquad w_0 \in \partial\mathcal{L} \tag{2.4}$$

(initially defined by (2.3) as a holomorphic function on a neighborhood of 0) extends to an entire function of z. But this is obviously incorrect because, putting $e^z = a$ in (2.4), we have in terms of a variable a a Laurent series with finite radii of convergence $0 < (1/R) < R < \infty$ and on each of the circles $\{|a| = 1/R, R\}$ function (2.4) must have a singularity.

Note. A perhaps more transparent and simple-minded counter-example could also be constructed by using a power series, e.g.,

$$v(z,w) := \int_{k,\ell=0}^{\infty} \frac{(k+\ell)!}{(2k+1)!\,\ell!} z^{2k+1} w^{\ell}. \tag{2.5}$$

Then, it is elementary to check that v is well-defined in the unit bidisk $\{|z| < 1, |w| < 1\}$ and satisfies (2.2) there. Also, $v(0,w) \equiv 0$ and, hence, by Fredholm's reasoning $v(z,1)$ must extend to be an entire function (of z), which is obviously false. In fact, (2.5) diverges for all values of $z \neq 0$ when $w = 1$.

Thus, what Fredholm attributes to Kovalevskaya is not true. Let us examine now what Kovalevskaya actually says.

3. Kovalevskaya's Theorem

In fact, Kovalevskaya [K, p.22] does not state a theorem at all, but merely looks at a few examples. The issue is the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial w} = \frac{\partial^2 u}{\partial z^2} & \text{with data} \\ u(z, w_0) = F(z) \end{cases} \tag{3.1}$$

where F is a given holomorphic function on some neighborhood of z_0.

Problem (3.1) is *characteristic* (cf. [Hö, Ch. 9]), because of the presence of the second derivative on the right. (If the right-hand side were of the form $A\frac{\partial u}{\partial z} + Bu + C$ with A, B, C holomorphic near z^0, the problem would be "correct" (non-characteristic) and (3.1) would have a (unique) holomorphic solution in a neighborhood of (z^0, w^0).) But due to the characteristicity, (3.1) will admit only a formal power series solution that, in general, converges nowhere. Kovalevskaya gives an example indicating that if F has a *finite* radius of convergence but is not entire, the formal solution to (3.1) will not converge on any neighborhood of w_0. This can be embodied in a rigorous theorem:

Theorem 1 (essentially Kovalevskaya). *If $u(z,w)$ is holomorphic on some neighborhood of (z_0, w_0), say on the bidisk $\triangle(z^0, w^0; r) := \{(z,w) \in \mathbb{C}^2 : |z - z^0| < r, |w - w^0| < r\}$ and satisfies* (3.1) *there, then u extends holomorphically to the "tube" $\Omega := \{(z,w) : z \in \mathbb{C}, |w - w^0| < r\}$.*

Note. Theorem 1 does not justify Fredholm's conclusion, because he only has holomorphy of u in a w-disk for *one fixed* z_0. There is no bidisk of regularity of u containing (z_0, w_0)!

Although a direct proof of Thm. 1 is not hard (see the following Thms.2, 2′), we prefer here to give a geometric proof based on a simple but extremely useful observation of Zerner [Hö, Thm. 9.4.7].

Proof. Without loss of generality, let $z_0 = w_0 = 0$. Fix $\varrho : 0 < \varrho < 1$ and consider a family of ellipsoids in $\mathbb{C}^2$ $(= \mathbb{R}^4)$

$$E_\lambda^\varrho = E_\lambda := \left\{(z,w) : \frac{|z|^2}{\lambda^2} + \frac{|w|^2}{(\varrho r)^2} < 1\right\},$$

where $\lambda > 0$, $\lambda \uparrow +\infty$ is a parameter. Clearly, the family $\{E_\lambda\}$ is increasing, $E_\lambda \subset \Omega$ for all λ and $\bigcup_{\lambda>0} E_\lambda^\varrho = \Omega^\varrho := \{(z,w) : z \in \mathbb{C}, |w| < \varrho r\}$. Also, $E_\lambda^\varrho \subset \triangle = \triangle(0,0;r)$ for all sufficiently small λ. It suffices to show that u extends to Ω^ϱ and then let $\varrho \uparrow 1$. The characteristic (with respect to (3.1)) points on the surfaces $\partial E_\lambda := \left\{(z,w) : \frac{z\bar{z}}{\lambda^2} + \frac{w\bar{w}}{(\varrho r)^2} - 1 =: \varphi(z,w) = 0\right\}$ i.e. those that satisfy $(\partial\varphi/\partial z)^2 = 0$, are (since $\partial\varphi/\partial z = \frac{\bar{z}}{\lambda^2}$), $\{(0, \varrho r e^{i\theta}) : 0 \le \theta \le 2\pi\}$. So for each λ the set $(\partial E_\lambda)_{\text{char}}$ of characteristic points of ∂E_λ is in $\triangle$, and hence u is holomorphic across $(\partial E_\lambda)_{\text{char}}$ for all $\lambda > 0$. By Zerner's theorem, u is holomorphically continuable across $\partial E_\lambda \setminus (\partial E_\lambda)_{\text{char}}$, provided it is holomorphic in E_λ (and, of course, satisfies (3.1) there). So, since ∂E_λ is compact, it follows that

$$\text{if } u \text{ is holomorphic in } E_\lambda, \text{ then it is also holomorphic in } E_{\lambda+\eta}, \text{ for sufficiently small } \eta > 0. \tag{3.2}$$

The rest of the argument is rather standard (cf., e.g., [KS], [J]). If we let $\mathcal{F}$ be the set $\mathcal{F} := \{\lambda > 0 : u \text{ is holomorphic in } E_\lambda\}$, then (3.2) implies that $\mathcal{F}$ is open, while it is obvious that $\mathcal{F}$ is closed ($\lambda_n \uparrow \lambda_0 \Rightarrow E_{\lambda_n} \uparrow E_{\lambda_0}$). $\mathcal{F} \neq \emptyset$ by the hypothesis ($E_\lambda \subset \triangle$ for small λ). So $\mathcal{F} = \mathbb{R}_+$ and u extends holomorphically to Ω^ϱ and hence to all of Ω.

Remark. The advantage of the above proof is obvious. Since it relies only upon characteristic directions for the operator (3.1), Thm. 1 in fact extends *mutatis mutandis* to all equations $\frac{\partial^q u}{\partial z^q} = \sum_{m+n \le q-1} a_{mn}(z,w) \frac{\partial^{m+n} u}{\partial z^m \partial w^n}$ whose coefficients a_{mn} are entire functions of (z,w).

Although, as we remarked above, Thm. 1 does not save Fredholm's proof, one can modify Fredholm's argument so as to obtain his conclusion and even more. First, we require the following stronger version of Thm. 1.

Theorem 2. *If $u(z,w)$ is holomorphic in the bidisk $\triangle = D_1 \times D_2$, $D_1 = \{z : |z - z_0| < \varrho_1\}$, $D_2 = \{w : |w - w_0| < \varrho_2\}$, satisfies* (3.1) *there and the function*

$w \mapsto u(z_0, w)$ extends holomorphically to a neighborhood of some point $w^* \in \partial\mathbb{D}_2$ (i.e., $|w - w_0| = \varrho_2$), then the even part (with respect to z_0) of $z \mapsto u(z, w_1)$ extends as an entire function (in fact, of order ≤ 2) for each $w_1 \in D_2$ sufficiently close to w^*.

(The function $z \mapsto u(z, w_1)$ is well-defined for z in D_1 as long as $w_1 \in D_2$.)

The even part of a power series about z_0 is defined by

$$(E_{z_0} f)(t) = \frac{1}{2}[f(z_0 + t) + f(z_0 - t)].$$

Note. Examples (2.3) and (2.5) do not contradict Thm. 2, since in both cases the even parts of those functions (with respect to $z_0 = 0$) vanish identically.

Clearly, the even part $E_{z_0} u(t, w) = \frac{1}{2}[u(z_0 + t, w) + u(z_0 - t, w)]$ of the function u satisfying the "heat equation" (3.1) satisfies the same equation. Hence, Thm. 2 is an immediate corollary of the following sharper theorem.

Theorem 2′. *Let $u(z, w)$ be holomorphic in the bidisk $\Delta = D_1 \times D_2$, where, as above, $D_1 = \{z \in \mathbb{C} : |z - z_0| < \varrho_1\}$, $D_2 = \{w \in \mathbb{C} : |w - w_0| < \varrho_2\}$, even with respect to z about the point z_0 (i.e., $u(z_0 + t, w) = u(z_0 - t, w)$ for small $|t|$) and satisfy in Δ the heat equation (3.1). Suppose $w \mapsto u(z_0, w)$ extends analytically across $w^* \in \partial D_2$. Then there are constants $\varrho, C, A > 0$ such that for all $w \in D_2 \cap \{w : |w - w^*| < \varrho\}$, the function $z \mapsto u(z, w)$ (which is holomorphic in D_1) admits the Cauchy majorant*

$$Ce^{A(z-z_0)^2}, \tag{3.3}$$

and hence extends as an entire function (of z) of order at most 2.

Remarks. (i) Recall that, given two power series

$$f(z) = \sum_0^\infty a_n (z - z_0)^n, \quad F(z) = \sum_0^\infty A_n (z - z_0)^n,$$

we say that F is a (Cauchy) majorant for f (with notation $f \ll F$) if $|a_n| \leq A_n, \forall n$. Note that then $f' \ll F'$, etc.

(ii) Although Thms. 2 and 2′ are not contained in Kovalevskaya's paper, the idea of the proof is implicit there. She does, in fact, point out (p.22) an example of "data" F in (3.1), an entire function of order 3, for which the formal power series solution of (3.1) is everywhere divergent.

Proof of Thm. 2′. Let r be the radius of the disk of holomorphy of $w \mapsto u(z_0, w)$ about w^*. So, as soon as $w \in D_2$ is close enough to w^*, say $|w - w^*| < \frac{1}{2}r$,

$w' \mapsto u(z_0, w')$ is holomorphic in a disk about w of radius $\frac{1}{2}r$. Then, there exist constants B, C dependent on r but not on w such that

$$|\partial_w^n u(z_0, w)| \le CB^n n!\,,$$

and since $\partial_w u = \partial_z^2 u$,

$$\left|\frac{\partial_z^{2n} u(z_0, w)}{(2n)!}\right| \le \frac{CB^n n!}{(2n)!} \le \frac{CB^n}{n!}$$

because $(n!)^2 \le (2n)!$. Since u is even with respect to z_0, the odd-order terms at $z = z_0$ simply vanish. Thus, the power series expansion of $z \mapsto u(z, w)$ about z_0 is majorized by

$$\sum_0^\infty C\frac{B^n}{n!}(z - z_0)^{2n} = Ce^{A(z-z_0)^2}, \qquad A = \sqrt{B}\,.$$

Remark. Again, the mistake of Fredholm (and Mittag-Leffler) was to assume that bounds for the *even-order* Taylor coefficients are valid also for the *odd-order* ones, which of course is false.

Theorem 1 can be deduced from Theorem 2 by means of the following purely function-theoretic lemma.

Lemma 1. *If $f(z)$ is holomorphic in a disk $\mathcal{D}(z_0, R)$ about z_0 of radius R and for all $z_1 : |z_1 - z_0| < \varrho$, where ϱ is a given positive number, its even part $E_{z_1} f(z)$ about z_1 is entire, then f itself is an entire function.*

Proof of lemma. Without loss of generality we can assume $z_0 = 0$, $\varrho < R/2$. Thus, for all $z_1 : |z_1| < \varrho$, $E_{z_1} f = E(z_1; z) = \frac{1}{2}(f(z+z_1) + f(z_1 - z))$ extends as an entire function (of z). By the Cauchy-Hadamard inequalities we have then

$$\left(\frac{\left|E_{z_1}^{(n)}(z_1; z)\right|_{z=z_1}}{n!}\right)^{1/n} = \left(\frac{\frac{1}{2}\left|f^{(n)}(2z_1) + (-1)^n f^{(n)}(0)\right|}{n!}\right)^{1/n}$$
$$\le \varepsilon_n(z_1), \qquad \text{where } \lim_{n\to\infty} \varepsilon_n(z_1) = 0\,. \tag{3.4}$$

Since f is analytic in $\mathcal{D}(0, R)$, we also have

$$\left|\frac{f^{(n)}(0)}{n!}\right|^{1/n} \le A^{1/n} R^{-1}\,, \tag{3.5}$$

where $A > 0$ is a constant. From (3.4), (3.5) we obtain

$$\frac{|f^{(n)}(2z_1)|}{n!} \leq 2\varepsilon_n^n(z_1) + AR^{-n}.$$

Therefore,

$$\left(\frac{|f^{(n)}(2z_1)|}{n!}\right)^{1/n} \leq 2^{1/n}\varepsilon_n(z_1) + A^{1/n}R^{-1}.$$

Thus,

$$\overline{\lim_{n\to\infty}} \left(\frac{|f^{(n)}(2z_1)|}{n!}\right)^{1/n} \leq \frac{1}{R}.$$

So, for all $z_1 : |z_1| < \varrho$, f extends as an analytic function to the disk $\{(z - 2z_1| < R\}$ about $2z_1$. Therefore, f obviously extends to the disk $\mathcal{D}(0, R + \varrho)$ about the origin. Iterating this argument we have f extended to all of $\mathbb{C}$.

Remarks. (i) By modifying the above argument, we can slightly weaken the hypothesis and assume that it holds for all z_1 in a subset of the second category (with respect to $\mathcal{D}(0, R)$). There is an open problem, however: *How big a set of z_1's with respect to which $(E_{z_1} f)$ is entire do we need in order to assert that f is entire?* It's easy to see that a finite set of collinear points in arithmetic progression is too small.

Example. Let g be holomorphic in the strip $|\operatorname{Im} z| < Y$, where Y is sufficiently large, and of period 2π, having a pole at iY and regular at $-iY$. Then $f(z) = g(z) - g(-z)$ furnishes the desired example.

(ii) In the situation of Kovalevskaya's Theorem (Thms. 1, 2, and 2′), we know more than that the E_{z_1} are entire; there is also an *order* restriction and it is easy to incorporate this into the proof of the lemma.

4. Fredholm's argument corrected.

We can now prove, using Thm. 2′, the following generalization of Fredholm's assertion.

Theorem 3. *If $0 < \overline{\lim}\, |a_n|^{1/n} < 1$, then $\sum_0^\infty a_n \zeta^{n^2}$ is not analytically continuable beyond the unit circle.*

Corollary. *If $a_n = a^n$, $0 < |a| < 1$, we obtain precisely the statement* (iii) *(see* §1*) of Fredholm's result.*

Proof. Consider

$$u(z, w) := \sum_0^\infty a_n (\cosh nz) e^{n^2 w}, \tag{4.1}$$

first in the domain $\mathcal{D}_1 \times \mathcal{L}$, where $\mathcal{L} = \{\mathrm{Re}\, w < 0\}$ and $\mathcal{D}_1$ is a disk around $z = 0$ in which $\sum_0^\infty |a_n|\,|\cosh nz|$ converges. Assuming that $\sum_0^\infty a_n e^{n^2 w} = u(0, w)$ continues across some point $w^* \in \partial\mathcal{L}$, we will derive a contradiction. Indeed, by Thm. 2′ we have that for all $w \in \mathcal{L}$ sufficiently near to w^* the function

$$z \mapsto u(z, w) = \sum_0^\infty \left(a_n e^{n^2 w}\right) \cosh nz \tag{4.2}$$

(which for $z \in \mathcal{D}_1$ is analytic, hence has a power series that converges for small $|z|$ and is *even* about $z = 0$), is $\ll Ce^{Az^2} =: F(z)$. Therefore, for $z \in \mathcal{D}_1$, and $w \in \mathcal{L}$ close to w^* we have

$$u(z, w) = \sum_0^\infty \alpha_n(w) z^n ,$$

where the coefficients $\alpha_n(w)$ satisfy

$$|\alpha_n(w)| = \left| \frac{\left(\frac{\partial}{\partial z}\right)^n u(0, w)}{n!} \right| \le \frac{F^{(n)}(0)}{n!} . \tag{4.3}$$

Now,

$$\alpha_n(w) = \sum_{m=0}^\infty a_m e^{m^2 w} b_m^n ,$$

where b_m^n are the Taylor coefficients of $\cosh mz$, i.e., $b_m^n = \begin{cases} m^n/n!, & n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$

Hence there are constants C, ϱ with $C > 0$, $0 < \varrho < 1$ such that

$$|a_m|\, b_m^n \le C\varrho^m m^n/n! .$$

We now let $w \to w^*$ from inside $\mathcal{L}$. The last estimate, together with the Lebesgue dominated convergence theorem imply that

$$|\alpha_n(w^*)| = \left| \sum_{m=0}^\infty a_m e^{m^2 w^*} b_m^n \right|$$

admits the same estimates (4.3). Hence,

$$z \mapsto u(z, w^*) = \sum_0^\infty (a_m e^{m^2 w^*}) \cosh mz$$

extends as an entire function to the whole z-plane (and, is even $\ll F$). But this is impossible, because it is a convergent Laurent series in the variable e^z with the *finite* convergence annulus $1/R < |e^z| < R$ for some $R > 1$. Thus, it has singularities on each of the bounding circles and that concludes the proof.

5. Further Remarks

(i) One could also prove Fredholm's theorem using Hadamard's "multiplication of singularities" theorem (D, p.346) (plus a few more things), but then the heat equation plays no role. Here are the details.

Suppose $\sum_{n=0}^{\infty} a^n \zeta^{n^2}$, $a \in \mathbb{C} \setminus \{0\}$ is continuable across the point $\zeta = \zeta_0$ of $\partial\mathbb{D}$. Setting

$$s(n) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\sum_{n=0}^{\infty} a^{\sqrt{n}} s(n) \zeta^n$$

is continuable across ζ_0. Now let $b \in \mathbb{C} \setminus \{0\}$. Then

$$\sum_0^{\infty} b^{\sqrt{n}} s(n) \zeta^n = \sum_0^{\infty} \lambda_n a^{\sqrt{n}} s(n) \zeta^n ,$$

where

$$\lambda_n = (b/a)^{\sqrt{n}} = e^{\sqrt{n}(\beta - \alpha)} \quad (b = e^{\beta}, a = e^{\alpha}) .$$

We have now

Lemma 2. *$\sum_{n=0}^{\infty} e^{\sqrt{n}\gamma} \zeta^n$ is, for every $\gamma \in \mathbb{C}$, analytically continuable across all points of $\partial\mathbb{D} \setminus \{1\}$.*

Assuming the lemma for the moment, we conclude by Hadamard's "multiplication of singularities" theorem that

$$\sum_0^{\infty} \lambda_n a^{\sqrt{n}} s(n) \zeta^n \left(= \sum_0^{\infty} b^{\sqrt{n}} s(n) \zeta^n = \sum_0^{\infty} b^n \zeta^{n^2} \right)$$

is continuable across ζ_0. In fact, there is a fixed neighborhood U of ζ_0 in $\mathbb{C}$ such that $\sum_0^{\infty} e^{nz} \zeta^{n^2}$ is, for every $z \in \mathbb{C}$, continuable into U. In particular, the function

$$z \mapsto \sum_0^{\infty} \zeta_0^{n^2} e^{nz} ,$$

defined initially for $\operatorname{Re} z < 0$, extends as an entire function. But we know that there must be a singularity on the line $\{\operatorname{Re} z = 0\}$. This contradiction completes the argument.

Proof of Lemma 2. Set

$$\varphi(t) = e^{\gamma\sqrt{t}} + e^{-\gamma\sqrt{t}};$$
$$\psi(t) = \frac{e^{\gamma\sqrt{t}} - e^{-\gamma\sqrt{t}}}{\sqrt{t}}.$$

Note that φ, ψ are *entire* functions of order $1/2$, so by Leau's theorem (see [D, p.336) or [B, Thm. 1.3.11]), the series

$$\sum_0^\infty \varphi(n) z^n, \quad \sum_0^\infty \psi(n) z^n$$

both extend analytically to $\mathbb{C} \setminus \{1\}$.

Claim. *The function $f(z) := \sum_1^\infty \sqrt{n} z^n$ is analytically continuable across $\partial\mathbb{D} \setminus \{1\}$.*

Assuming the claim, note that then by Hadamard's theorem the Hadamard product of $\sum \sqrt{n} z^n$ with $\sum \psi(n) z^n$, i.e., $\sum \theta(n) z^n$, where $\theta(n) := e^{\gamma\sqrt{n}} - e^{-\gamma\sqrt{n}}$ is also continuable across $\partial\mathbb{D} \setminus \{1\}$. Hence, finally,

$$2\sum_0^\infty e^{\gamma\sqrt{n}} z^n = \sum_0^\infty (\varphi(n) + \theta(n)) z^n$$

also extends analytically across $\partial\mathbb{D} \setminus \{1\}$.

The claim above is a direct corollary of the theorem of Le Roy and Lindelöf [D, p.340], but a straightforward (simple) proof can also be given. It suffices to check that

$$h(z) = \sum_1^\infty n^{-1/2} z^n, \qquad |z| < 1$$

is continuable across $\partial\mathbb{D} \setminus \{1\}$, since $f(z) = zh'(z)$. The change of variables $\ln t = u/n$, shows that

$$n^{-1/2} = \frac{1}{\Gamma(1/2)} \int_1^\infty (\ln t)^{-1/2} t^{-n-1}\, dt.$$

Hence, for $z : |z| < 1$ we have

$$h(z) = \frac{1}{\Gamma(1/2)} \int_1^\infty \sum_1^\infty \frac{z^n (\ln t)^{-1/2}}{t^{n+1}} dt$$

$$= \frac{z}{\Gamma(1/2)} \int_1^\infty \frac{(\ln t)^{-1/2}}{t(t-z)} dt \qquad (5.1)$$

and the Claim follows. (A more careful analysis of (5.1) yields that the Cauchy integral actually extends analytically across all points on $(1, \infty)$ except 1; so $\sum n^{1/2} z^n$ extends as a branch of a multiple-valued analytic function to all of $\mathbb{C} \setminus \{1\}$.)

(ii) Using Hadamard's "multiplication of singularities" theorem and Lemma 2 we could slightly weaken the hypotheses of Thm. 3, assuming merely $0 < \overline{\lim} |a_n|^{1/n} < \infty$. Indeed, introducing a compensating factor p^n into (4.1), for $p > 0$ sufficiently small, so that $\sum_0^\infty a_n e^{n^2 w^*} p^n (\cosh nz)$ is defined in a neighborhood of $z = 0$, and applying Thm. 3 we obtain that $\zeta \mapsto \sum_0^\infty a_n p^n \zeta^{n^2}$ is nowhere continuable across $\partial\mathbb{D}$. On the other hand, by Lemma 2, $\sum_0^\infty p^{\sqrt{n}} \zeta^n = \sum_0^\infty e^{\log p \cdot \sqrt{n}} \zeta^n$ extends across $\partial\mathbb{D} \setminus \{1\}$. Set $b_n = \begin{cases} a_{\sqrt{n}}, & n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$. Then, if $\sum_0^\infty a_n \zeta^{n^2} = \sum_0^\infty b_n \zeta^n$ extends across $\zeta_0 \in \partial\mathbb{D}$ in view of Hadamard's theorem so does

$$\sum_0^\infty a_n p^n \zeta^{n^2} = \sum_0^\infty b_n p^{\sqrt{n}} \zeta^n ,$$

and we arrive at a contradiction.

(iii) We have used repeatedly the fact that a Dirichlet series of the special form $\sum_{n=0}^\infty a_n e^{nw}$ has a singularity on its abscissa of convergence, which is just another form of the corresponding principle for the power series ($\zeta = e^w$). For a more general Dirichlet series $\sum a_n e^{\lambda_n w}$, this is no longer true, e.g.,

$$\sum_{n=1}^\infty \frac{(-1)^n}{n^w} = \sum_{n=1}^\infty (-1)^n e^{-w \log n} = \left(\frac{1}{2^{w-1}} - 1 \right) \zeta(w) ,$$

where $\zeta(w)$ is Riemann's ζ-function, obviously diverges for $\operatorname{Re} w \leq 0$, yet it represents an entire function (cf. [T]). In this regard, it is instructive to see why we *cannot* prove (the indeed *false* proposition) that $\zeta \mapsto \sum_1^\infty e^{\sqrt{n} z} \zeta^n$ is nowhere continuable across $\partial\mathbb{D}$ (in the ζ-plane) by the (corrected) Fredholm method.

$$v(z,w) := \sum_1^\infty e^{\sqrt{n}z+nw}, \quad \zeta = e^w$$

satisfies $\frac{\partial v}{\partial w} = \frac{\partial^2 v}{\partial z^2}$ in $\mathbb{C} \times \mathcal{L}$. If continuable across $w = w_0 \in \partial\mathcal{L}$, then $\sum e^{nw_0} e^{\sqrt{n}z}$ would have to be entire in z, and for w_0 with $\operatorname{Re} w_0 = 0$, $\operatorname{Im} w_0 \neq 2\pi ik$, $k \in \mathbb{Z}$ that is precisely what happens (cf. Lemma 2 and replace the parameter γ with variable z)! Even though this Dirichlet series cannot converge when $\operatorname{Re} z \geq 0$, the sum of the series (where $\operatorname{Re} z < 0$) is indeed continuable without singularities to the whole z-plane.

(iv) Professor L. Gårding has kindly pointed out to us another way of saving Fredholm's original theorem, which is worth sketching here.

Assuming, for just *one* $a_0 \neq 0$, $\sum_0^\infty a_0^n \zeta^{n^2}$ is continuable across $\zeta_0 \in \partial\mathbb{D}$, that is (again, put $a = e^z$, $\zeta = e^w$), $\sum e^{nz} e^{n^2 w}$ extends across $w^* = it_0$, $t_0 \in \mathbb{R}$. Then, by our argument (cf. Thms. 2′ and 3) if follows that the even-order derivatives with respect to z admit at $z = z_0$ (where $e^{z_0} = a_0$) "good" bounds:

$$\left| \sum_{n=1}^\infty n^{2k} a_0^n e^{n^2 it_0} \right| \leq CA^k \cdot k!\,,$$

where A, C are constants. If we write $c_{n,k} := n^{2k} e^{n^2 it_0} a_0^n$, the above inequality can be rewritten as

$$\left| \sum_{n=1}^\infty c_{n,k} \right| \leq CA^k \cdot k!\,. \tag{5.2}$$

The corresponding expression for the point $a = ra_0$, with $0 < r < 1$, is obviously $\sum_{n=1}^\infty c_{n,k} r^n$. Now, using Abel's partial summation formula it follows from (5.2) that

$$\left| \sum_1^\infty c_{n,k} r^n \right| \leq BA^k \cdot k!$$

for some new constant B, independent of k. Thus, the "right" estimates for the even-order derivatives (with respect to z) do hold on an *interval* of z-values, and, it is well-known that one obtains then for z interior to this interval, similar estimates for the odd-order derivatives and can conclude that $\sum e^{nz} e^{n^2 w^*}$ must be entire in z.

6. Epilogue

So far as we know, no one ever tried to follow up Fredholm's idea, and in fact, it is not at all clear that one can get much more out of it. On the other hand, J.F. Ritt [R] in 1917 proved Fabry's gap theorem for the Dirichlet series $f(z) := \sum a_n e^{\lambda_n z}$, where λ_n are positive integers such that $\lambda_n/n \to +\infty$, by showing that a function f so defined (even in the case of complex exponents subjected to some additional constraints) satisfies an infinite order ordinary differential equation of the type $g(D)f = 0$ $(D = d/dz)$, where g is an entire function of zero exponential type. He showed that any such f, continued along all paths, remains single-valued, and, moreover, its domain of existence coincides with the domain of absolute convergence of the defining series and is convex. Thus, if $f(z)$ has a finite abscissa of convergence c (i.e., converges for $z : \operatorname{Re} z < c$ and diverges for $z : \operatorname{Re} > c$), then in case it "breaks through" analytically somewhere across the line $\{z : \operatorname{Re} z = c\}$, it must be analytic in a larger half-plane $\{z : \operatorname{Re} z < c - \varepsilon, e > 0\}$ and hence converge there absolutely, which is an obvious contradiction. Ritt had, in fact, a few superfluous assumptions which were removed later by Valiron [V] and Polya [P]. (Also, cf. [H].)

Later on, apparently independently, L. Ehrenpreis [E] (1970) developed an approach to non-continuability of lacunary Dirichlet series also based on *infinite order ordinary differential equations* and *convolution equations*, rather than partial differential equations. This method is quite different from Fredholm's, insofar as Fredholm introduces a second independent variable (and a partial differential equation involving those two independent variables), whereas Ritt, Ehrenpreis, and their followers keep only one independent variable. Still, the methods seem to have enough in common to make it reasonable to juxtapose them. (Variants and extensions of Ehrenpreis' method were given later by Kawai, Berenstein, Struppa and others, cf. e.g., [BS1–3], [Ka].) Ehrenpreis' method can be motivated by looking at the Weierstrass series $\sum_0^\infty a_n \zeta^{2^n} = f(\zeta)$. We shall again put $\zeta = e^w$. Assuming that the power series has a radius of convergence 1, we get a Dirichlet series $\sum_0^\infty a_n e^{2^n w} =: \varphi(w)$, say, defined and holomorphic for $w \in \mathcal{L} := \{\operatorname{Re} w < 0\}$. Now, φ satisfies in $\mathcal{L}$ a myriad of functional equations. Indeed, if t is any *positive dyadic submultiple* of π, i.e., $t = \pi/2^k$, we have

$$\varphi(w + it) - \varphi(w) = P_t(w)\,, \tag{6.1}$$

where P_t is a polynomial in w. From this, as in Ritt's situation, it is clear that if φ were continuable across any point of $\partial\mathcal{L}$, it would extend analytically to a half plane $\{\operatorname{Re} w < \sigma\}$ for some $\sigma > 0$. Indeed, once φ "breaks through" $\partial\mathcal{L}$ somewhere horizontally, (6.1) enables us to complete the continuation vertically.

Let us now consider the more general Dirichlet series

$$\varphi(w) = \sum_0^\infty a_n e^{\lambda_n w}\,,$$

where, say, λ_n are positive integers with $\lim \lambda_n = \infty$. So, φ is essentially a power series, and hence (i) its abscissae of *convergence* and *absolute convergence* coincide (see, e.g., [T]); (ii) there is at least one singularity on its abscissa of convergence. To fix the ideas, let us assume that the abscissa of convergence is $\{\operatorname{Re} w = 0\}$. Suppose now that the sequence $\{\lambda_n\}$ satisfies the following property:

(P) For every real positive t (or even: a sequence $\{t_j\}, t_j \downarrow 0$) and every $\varrho > 0$, there is a measure μ supported in $\{|z| \leq \varrho\}$ such that

$$e^{i\lambda t} = \int e^{i\lambda z} d\mu(z) \tag{6.2}$$

holds for all $\lambda \in \{\lambda_n\}$.

(This is an *interpolation* property. Clearly the "fewer" λ_n's there are, the better the chances that $\{\lambda_n\}$ satisfies (P).) If (6.2) holds, then for every function of the type

$$F(w) = \sum_1^N c_n e^{\lambda_n w}, \qquad c_n \in \mathbb{C}, \quad N\text{-arbitrary},$$

we have

$$F(w + it) = \sum_1^N c_n e^{\lambda_n (w+it)}$$

and

$$\int F(w + z) d\mu(z) = \sum_1^N c_n \int e^{\lambda_n (w+z)} d\mu(z),$$

and so, in view of (6.2),

$$F(w + it) = \int F(w + z) d\mu(z),$$

where μ depends on t and ϱ but not on F! Now, $\varphi = \sum_0^\infty a_n e^{\lambda_n w}$ can be uniformly approximated on compact subsets of $\mathcal{L}$ by functions like F, so we have

Proposition. *If $\{\lambda_n\}$ satisfies (P), then every Dirichlet series $\varphi(w) = \sum_0^\infty a_n e^{\lambda_n w}$ converging in $\mathcal{L}$ satisfies, for each $t > 0$ and $\varrho > 0$ a functional equation*

$$\varphi(w + it) = \int \varphi(w + z) d\mu(z),$$

with a measure μ supported in $\{|z| \leq \varrho\}$.

Consequently, it follows just as with the Weierstrass series:

Corollary. *If $\{\lambda_n\}$ have property (P), a Dirichlet series $\sum_0^\infty a_n e^{\lambda_n w}$ cannot be continuable across any point of its abscissa of convergence.*

Now, it is well-known (cf. [B, §1]), that every entire function of exponential type zero (the so-called minimal type) has a Borel representation as $\int e^{iz\lambda} d\mu(z)$ with supp μ in an arbitrarily small neighborhood of $z = 0$. Since the sequence $\{e^{i\lambda_n t}\}$ is bounded, we have the following.

Theorem 4. *Suppose $\{\lambda_n\} \subset \mathbb{N}$ are such that for every bounded complex sequence $\{\alpha_n\}$, there is an entire function of minimal exponential type $\psi(\lambda)$, satisfying $\psi(\lambda_n) = \alpha_n$, $n = 1, 2, \ldots$. Then, no Dirichlet series $\sum_0^\infty a_n e^{\lambda_n w}$ can be continued analytically across any point of its abscissa of convergence.*

Ideas like the above can be used to deduce the Fabry gap theorem and some generalizations, as Ehrenpreis [E] has shown. Let us content ourselves here with deducing Fredholm's theorem. Obviously, we will again get a result *stronger* than Fredholm's dealing with *all* series $\sum_0^\infty a_n \zeta^{n^2}$ having a radius of convergence 1, not merely $a_n = a^n$, if we show that the *sequence of positive squares has property* (P), or even more, that we can solve

$$\varphi(n^2) = \alpha_n, \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\}_0^\infty$ is a given *bounded* sequence and φ is of zero type.

Here is a simple *ad hoc* proof. Extend the definition of α_n to all integral n by evenness.

Let

$$g(t) := \sum_{-\infty}^{\infty} (1 + n^2)^{-1} \alpha_n e^{int}.$$

Then g is bounded and continuous on $\mathbb{R}$, even and of period 2π, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt = (1 + n^2)^{-1} \alpha_n, \qquad n \in \mathbb{Z}.$$

Thus,

$$G(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-itz} dt$$

is an even entire function of exponential type $\leq \pi$, satisfying

$$G(n) = (1 + n^2)^{-1} \alpha_n, \qquad n \in \mathbb{Z}.$$

We may write $G(z) = H(z^2)$ where H is entire, of order $\leq 1/2$, and then

$$(n^2+1)H(n^2) = \alpha_n, \qquad n \in \mathbb{Z},$$

so $F(z) := (z+1)H(z)$ is entire, of order $\leq 1/2$ and solves the interpolation problem $F(n^2) = \alpha_n$.

In summary, the basic idea of Thm. 4 is that non-continuability of a power series is reduced to a *dual problem of interpolation by entire functions of small type.* For such interpolation problems, there is an imposing literature (cf., e.g., [Bo]).

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Department of Mathematics
University of Arkansas
Fayetteville, AR 72701

Department of Mathematics
Royal Institute of Technology
S-100 44 Stockholm